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Modification of the adaptive bidimensional kernel density estimation of well-being distribution and poverty index using the arithmetic mean

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ABSTRACT
In this paper, we propose a bi-dimensional extension of the Foster, Greer, and Thorbecke index using the adaptive kernel density. The estimator we proposed is based on Riemann sums and on the adaptive kernel density.
The new estimator is constructed with adaptive bi-dimensional kernel and the bandwith of the new improvement is obtained depending on the arithmetic mean of the observations. Simulated example is presented, including comparisons with three others estimators. The performance of the proposed new estimator is evaluated via the variance and the mean square error criterion. The results of the simulation study were very promising; it shows that our modified estimator performs well in all cases.

KEYWORDS
poverty line, bi-dimensional extension, modified adaptive bi-dimensional kernel estimator, Riemann sums, arithmetic mean of the observations.

1. Introduction

[3] introduce a bi-dimensional extension of the Foster, Greer, and Thorbecke index [4]. Later, [2] constructed the kernel-type version of the bi-dimensional extension of the Foster, Greer, and Thorbecke index for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ and estabilished uniform consistancies of this estimator. [1] used the adaptive bi-dimensional kernel density and Riemann sums to construct an estiamtor P_n^{λ} and estabilished that it was almost-sure uniform consistant and uniform mean square convergent.
To make ideas more clear, let x and y be two indicators of individual well-being among,

for example, income, expenditures, caloric consumption, life expectancy, height, body weight, extent of personal safety and freedom, etc. Throughout this papaer (X, Y) stands for the value of (x, y) for a randomly selected individual of the population. Then (X, Y) is a random couple of nonnegative real numbers defined of a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose cumulative distribution function (cdf) is denoted by $F(.,.)$ and

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we suppose that it admits a *pdf* $f(.,.)$. From now all the expections are done with respect to the probability space.
The bi-dimensional extension of the FGT [4] class of poverty by DSY [3] is denoted by $P(z_1, z_2, \alpha_1, \alpha_2)$ and is defined as follows, for $(\alpha_1 \geq 0, \alpha_2 \geq 0)$:

$$P(z_1, z_2, \alpha_1, \alpha_2) = \begin{cases} \int_0^{z_1} \int_0^{z_2} \left(\frac{z_1 - x}{z_1}\right)^{\alpha_1} \left(\frac{z_2 - y}{z_2}\right)^{\alpha_2} f(x, y) dx dy & \text{if} z_1 > 0 \text{ and } z_2 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where z_1 (resp. z_2) represents the poverty line for the dimension x (resp. y). This index is useful for ordinal robust comparisons of poverty, even when the measurements are made across the intersection of the two dimensions considered. The parameters α_1 and α_2 capture the aversion to inequality in poverty in the x and in the y dimensions, respectively.

Now let us consider, for $n \geq 1$, a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ from (X, Y), defined of the probability space defined above.
The empirical plug-in estimator of (1) is given by

$$\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 - \frac{X_i}{z_1}\right)_+^{\alpha_1} \left(1 - \frac{Y_j}{z_2}\right)_+^{\alpha_2}$$

where $x_+ = max(0, x)$. [2] propose new kernels estimators, based on the Riemann sum.

1.1. *Bi-dimensional Kernel type estimator*

[2] consider the classical estimator of the density $f(x, y)$:

$$\widehat{f}(x, y) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h_1 h_2} K\left(\frac{x - X_i}{h_1}\right) K\left(\frac{y - Y_i}{h_2}\right)$$

and propose a kernel estimator given by

$$P_n(z_1, z_2, \alpha_1, \alpha_2) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i=0}^{\left[\frac{z_1}{h_1}\right]} \sum_{j=0}^{\left[\frac{z_2}{h_2}\right]} \left(1 - \frac{ih_1}{z_1}\right)^{\alpha_1} \left(1 - \frac{ih_2}{z_2}\right)^{\alpha_2} K\left(\frac{X_k - ih_1}{h_1}\right) K\left(\frac{Y_k - ih_2}{h_2}\right),$$

$K(x)$ is a Borel function satisfying the following hypotheses:

$$(\mathbf{H}_1) \quad \underset{-\infty < x < +\infty}{sup} |K(x)| < +\infty, \quad (\mathbf{H}_2) \quad \int_{\mathbb{R}} K(x) dx = 1, \quad (\mathbf{H}_3) \quad \lim_{x \to \pm\infty} |x||K(x)| = 0.$$

The kernel method requires a prior choice of the smoothing parameters h_1 and h_2, which are then set at any point where the distribution is estimated. This constraint finds its limits when the concentration of the data is particularly heterogeneous in the sample.

To answer this problem, [1] considered the adaptive bi-dimensional kernel estimator of the density $f(x,y)$

$$\widehat{f_n}(x,y) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_1 h_2 \lambda_i} K\left(\frac{x-X_i}{h_1\lambda_i}\right) \frac{1}{h_2\lambda_i} K\left(\frac{y-Y_i}{h_2\lambda_i}\right). \tag{2}$$

1.2. *Adaptive bi-dimensional kernel type estimator*

Combining this two last facts, and based on Riemann sum, [1] are able to propose the following adaptive kernel estimator of the DSY index:

$$\begin{aligned} P_n^{\lambda}(z_1, z_2, \alpha_1, \alpha_2) & \\ &= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{\left[\frac{z_1}{h_1\lambda_k}\right]}\sum_{j=0}^{\left[\frac{z_2}{h_2\lambda_k}\right]} \left(1-\frac{ih_1\lambda_k}{z_1}\right)^{\alpha_1}\left(1-\frac{ih_2\lambda_k}{z_2}\right)^{\alpha_2} \\ &\quad \times K\left(\frac{X_k - ih_1\lambda_k}{h_1\lambda_k}\right) K\left(\frac{Y_k - ih_2\lambda_k}{h_2\lambda_k}\right), \end{aligned} \tag{3}$$

where $\left[\frac{\cdot}{h_i\lambda_k}\right]$ is the integer part of $\frac{\cdot}{h_i\lambda_k}$, $i=1,2$ $\alpha_i \geq 0$, h_i, $i=1,2$ are global fixed bandwidths that control the overall degree of smoothing. λ_k is a local bandwidth factor of each sample point (X_k, Y_k) that adapts to the density of the data. The local bandwidth factors are proportional to the square root of the underlying density functions at the sample points. As [6] puts it on page 101, "An obvious practical problem is deciding in the first place whether or not an observation is in a region of low density".

$$\lambda_k = \lambda(x_k, y_k) = \left[\frac{g}{\widehat{f}(x_k, y_k)}\right]^{\beta}.$$

Where g is the geometric mean of the $\hat{f}(X_k, Y_k)$ and β is the sensitivity parameter satisfying $0 \leq \beta \leq 1$.
They needed to derive the following one $\mathbb{K}$, from the function K, defined on $\mathbb{R}^2$ by $\mathbb{K}(x,y) = K(x)K(y)$. This latter inherits from K these three proporties:

$$\sup_{-\infty \leq x \leq +\infty} |\mathbb{K}(x,y)| < +\infty \text{ and } \int\int_{\mathbb{R}^2} \mathbb{K}(x,y)dxdy = 1, \quad \lim_{||(x,y)||\to\infty} ||(x,y)|||\mathbb{K}(x,y)| = 0.$$

Now additional hypotheses on $\mathbb{K}$ or K are the following:
($\mathbf{H}_4$) K is a bounded variation function on $\mathbb{R}$ and we denote by $V(\mathbb{R})$ its total variation.
($\mathbf{H}_5$) $\int\int_{\mathbb{R}^2} |uv||\mathbb{K}(u,v)|dudv < +\infty$.
($\mathbf{H}_6$) There exists a non increasing function λ such that $\lambda(\frac{u}{h_1\lambda_k}, \frac{v}{h_2\lambda_k}) = O(h_1 h_2 \lambda_k^2)$ on any bounded rectangle and for two couples of real numbers $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$|\mathbb{K}(x) - \mathbb{K}(y)| \leq \lambda||x-y|| \text{ and } \lambda(u,v) \to 0, (u,v) \to (0,0), u \geq 0, v \geq 0,$$

where $||.||$ stands for the Euclidean norm.
Finally, these conditions depend of the *pdf* $f(x, y)$:
$\mathbf{C}_1$**:** $f(x, y)$ is uniformly continuous.
$\mathbf{C}_2$**:** $f(x, y)$ admits almost everywhere derivative $f'(x, y) \in L_1(\mathbb{R} \times \mathbb{R})$
to establish the uniform almost sure consistency and the uniform mean square consistenty of (3).

1.3. *Modified adaptive bi-dimensional kernel estimator*

In this part of the paper, we modify the adaptive bi-dimensional kernel estimator, our modification base on using the arithmetic mean instead of the geometric mean when compting the local bandwith factor which is given as

$$\lambda_k^* = \lambda(x_k, y_k) = \Big[\frac{g}{\widehat{f}(x_k, y_k)}\Big]^{\beta}.$$

Where g is the arithmetic mean of the $\hat{f}(X_k, Y_k)$.
Then, the adaptive bi-dimensional kernel estimator is defined by (3) with replacing λ_k by λ_k^*. We use the arithmetic mean since its value is greater than or equal to the geometric mean, and that makes the value of the λ_k^* greater than λ_k. By maximizing the value of the local bandwidth factor, the value of the bandwidth will increase too; and this will enhance the performance of the adaptive bi-dimensional kernel estimator. The rest of the paper is organized as follows. In Section 2, we will provide simulation studies outcomes. In section 3, we will relivant comments as well as a comparison with results from the empirical approach and from the classical estimator and the adaptive bi-dimensional kernel estimator that were used until now.

2. Simulations

2.1. *Proceedings*

We conduct simulations based on 75 replications of samples of size $n = 100$ and $n = 1000$. We compute three estimators of the FGT index in order to compare them. We use for distribution, a couple (X, Y) of two independent coordinates with X following a Pareto distribution type on [0, 1] with parameters $x_0 = 0.02$ and $b = 0.2$, Y following a exponential distribution with parameter $\lambda = 1$. For the kernel, we use the Gaussian one that fullfil H_i, $i = 1, 2, \ldots, 6$. With a given fixed couple (z_1, z_2) poverty lines we compute our estimator $P_n^{\lambda^*}(z_1, z_2, \alpha_1, \alpha_2)$ based on the modified adaptive kernel density (**A2DKA**),$P_n^{\lambda}(z_1, z_2, \alpha_1, \alpha_2)$ based on the adaptive kernel density (**A2DK**), $P_n(z_1, z_2, \alpha_1, \alpha_2)$, with $h_j = 1/\sqrt{nlogn}$ $j = 1, 2$, based on the Parzen kernel density (**2DK**), and $\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2)$, the empirical plug-in estimator (**2DPE**).
We consider the respective sequences:

$$(P_{n,1}^{\lambda}{}^*(z_1, z_2, \alpha_1, \alpha_2), P_{n,2}^{\lambda}{}^*(z_1, z_2, \alpha_1, \alpha_2), \ldots, P_{n,75}^{\lambda}{}^*(z_1, z_2, \alpha_1, \alpha_2)),$$

$$(P_{n,1}^{\lambda}(z_1, z_2, \alpha_1, \alpha_2), P_{n,2}^{\lambda}(z_1, z_2, \alpha_1, \alpha_2), \ldots, P_{n,75}^{\lambda}(z_1, z_2, \alpha_1, \alpha_2)),$$

$$(P_{n,1}(z_1,z_2,\alpha_1,\alpha_2), P_{n,2}(z_1,z_2,\alpha_1,\alpha_2), \ldots, P_{n,75}(z_1,z_2,\alpha_1,\alpha_2))$$

and

$$(\widehat{P}_{n,1}(z_1,z_2,\alpha_1,\alpha_2), \widehat{P}_{n,2}(z_1,z_2,\alpha_1,\alpha_2), \ldots, \widehat{P}_{n,75}(z_1,z_2,\alpha_1,\alpha_2)).$$

For each sequence, we compute the mean value mv_i, the mean squared error MSE_i, and the standard deviation σ_i, with $i = 1$ corespond to our estimator (**A2DKA**), $i = 2$ to the adaptive kernel estimator (**A2DK**), $i = 3$ to the estimator based on the Parzen kernel density (**2DK**) and $i = 4$ to the empirical plug-in estimator (**2DPE**). For the first estimator, we have:

$$mv_1 = \overline{P_n^{\lambda *}(z_1,z_2,\alpha_1,\alpha_2)} = \frac{1}{75}\sum_{i=1}^{75} {P_{n,i}^{\lambda}}^*(z_1,z_2,\alpha_1,\alpha_2)$$

$$MSE_1 = \frac{1}{75}\sum_{i=1}^{75}({P_{n,i}^{\lambda}}^*(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2))^2$$

$$\sigma_1 = \frac{1}{74}\sum_{i=1}^{75}({P_{n,i}^{\lambda}}^*(z_1,z_2,\alpha_1,\alpha_2) - \overline{P_n^{\lambda *}(z_1,z_2,\alpha_1,\alpha_2)})^2.$$

For the three others cases, we define similarly mv_2, MSE_2 and σ_2; mv_3, MSE_3 and σ_3; mv_4, MSE_4 and σ_4.
Next, we choose $(\alpha_1,\alpha_2) = (0,0)$, $(\alpha_1,\alpha_2) = (1,1)$ and $(\alpha_1,\alpha_2) = (2,2)$. The outcomes are summerized in the table bellow.

2.2. *Table of outcomes*

The simulations below are conducted on 75 replications of samples of size $n = 100$.

The simulations below are conducted on 75 replications of samples of size $n = 1000$.

A comparison of simulation results in Tables 1 and 2 shows that for small and big samples, each point (z_1,z_2), our modified adaptive bi-dimensional kernel estimator (**A2DKA**) provides a much lower standard deviation and a much lower mean squared error leading to a small bias for the three values of α_1, α_2 considered. Thus, we can conclude that our modified bi-dimensional kernel estimator is recommended for small and big samples.

The graphs of the FGT class of poverty index measures (**2DPE**) and the estimators which are computed based on the sample of size 1000 are respectively presented in Figures 1,2 and 3 for $(\alpha_1 = 0, \alpha_2 = 0)$, $(\alpha_1 = 1, \alpha_2 = 1)$ and $(\alpha_1 = 2, \alpha_2 = 2)$.

(z_1, z_2)	(0.1,0.4)	(0.1,0.8)	(0.4,0.4)	(0.4,0.8)
(α_1, α_2)=(0,0), n=100				
MSE$_1$	0.001017477	0.001579771	0.00162765	0.001556037
MSE$_2$	0.002070277	0.001699959	0.001650602	0.001797785
MSE$_3$	0.001960097	0.001626364	0.001581319	0.002043911
MSE$_4$	131.3076	131.2884	131.3008	131.3269
σ_1	0.04521881	0.04001399	0.04061583	0.04218729
σ_2	0.0458067	0.04150821	0.0409012	0.04268583
σ_3	0.04457112	0.04059978	0.04003359	0.04551408
σ_4	11.53612	11.53528	11.53582	11.53697
(α_1, α_2)=(1,1),n=100				
MSE$_1$	0.001632151	0.001554977	0.001306821	0.001586957
MSE$_2$	0.001764191	0.001637264	0.001411491	0.001723984
MSE$_3$	0.001713138	0.001566595	0.001308023	0.001588923
MSE$_4$	0.0959773	0.09216794	0.350411	1.736491
σ_1	0.04067194	0.03969874	0.03639341	0.0401049
σ_2	0.04228512	0.0407356	0.03782281	0.04180049
σ_3	0.0416688	0.03984678	0.03641015	0.04012973
σ_4	0.3118883	0.3056361	0.5959415	1.326634
(α_1, α_2)=(2,2),n=100				
MSE$_1$	0.001882595	0.002048177	0.001614685	0.001399647
MSE$_2$	0.001981823	0.002081718	0.001728104	0.001614202
MSE$_3$	0.001973286	0.0020491	0.001631086	0.001486598
MSE$_4$	0.01901598	0.03354909	0.1624404	0.6601918
σ_1	0.04368106	0.04589401	0.04045374	0.04026493
σ_2	0.04481746	0.0459331	0.04185041	0.04044768
σ_3	0.04472082	0.04557182	0.04065867	0.03881607
σ_4	0.1388271	0.1843976	0.405753	0.8179935

Table 1. Comparative table of results simulations $n = 100$

3. Results

From the figures, it is clear that for all values of the poverty line (z_1, z_2), the performance of the modified estimator (**A2DKR**) is superior the empirical (**2DPE**), the classical kernel (**2DK**), the adaptive kernel (**A2DK**) and the modified adaptive kernel (**A2DKRA**) estimators. Also, the figures show that generally the performance of all the studied estimators becomes better by increasing the value of (α_1, α_2).

A comparison of simulation results in Tables 1 and 2 shows that for small and big samples, each point (z_1, z_2), our modified adaptive bi-dimensional kernel estimator (**A2DKR**) provides a much lower standard deviation and a much lower mean squared error leading to a small bias for the three values of (α_1, α_2) considered. Thus, we can conclude that our modified bi-dimensional kernel estimator is recommended for small and big samples.

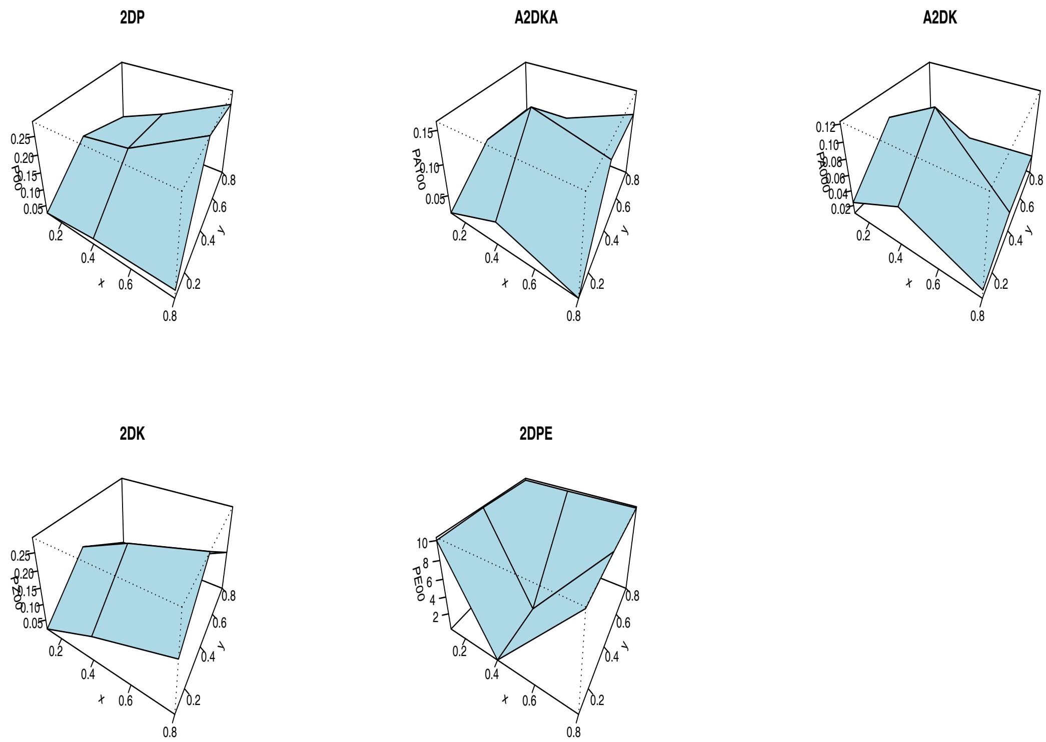

Figure 1. The FGT (2DP) poverty mesure and the estimators using $(\alpha_1 = 0, \alpha_2 = 0)$

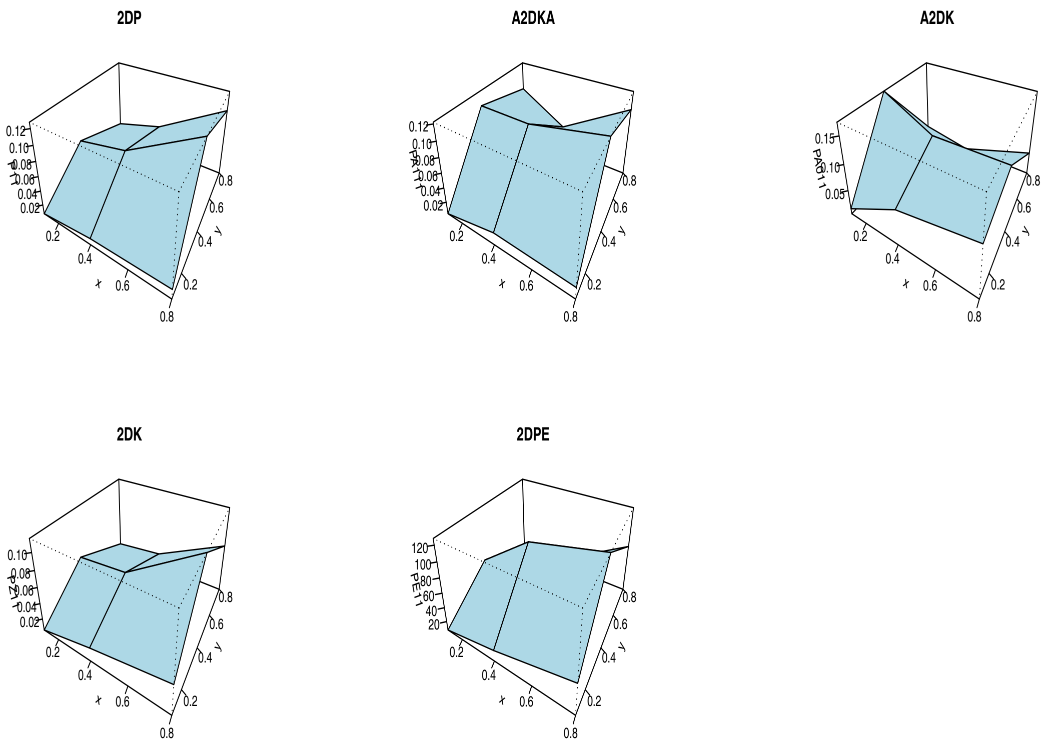

Figure 2. The FGT (2DP) poverty mesure and the estimators using $(\alpha_1 = 1, \alpha_2 = 1)$

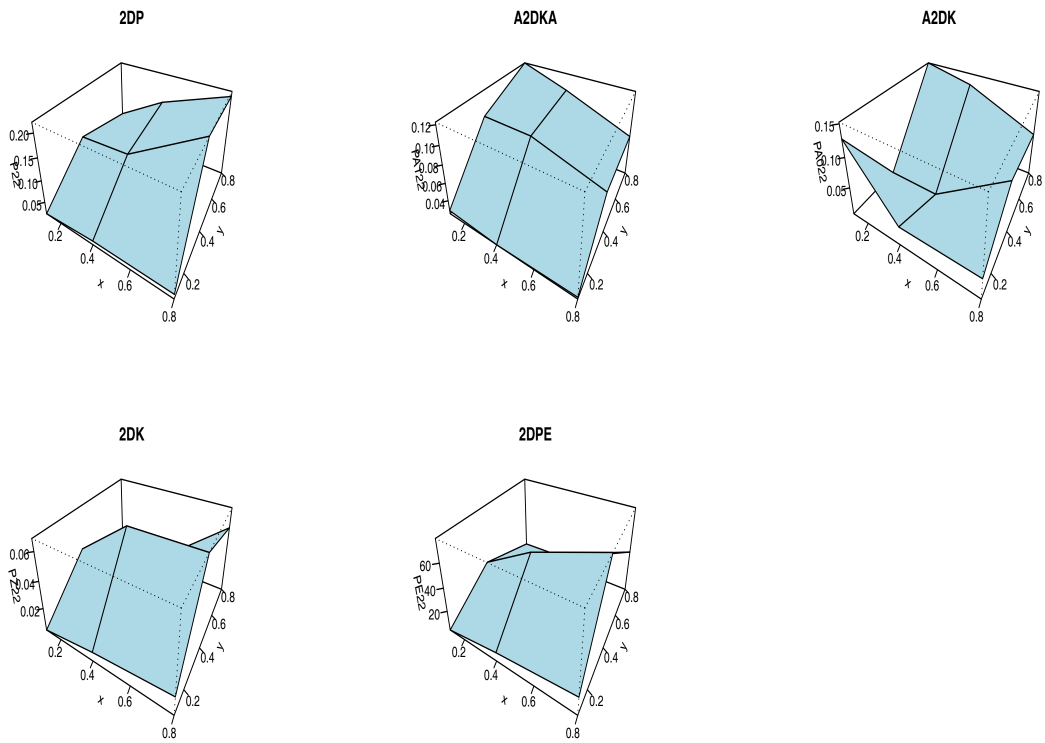

Figure 3. The FGT (2DP) poverty mesure and the estimators using $(\alpha_1 = 2, \alpha_2 = 2)$

(z_1, z_2)	(0.1,0.4)	(0.1,0.8)	(0.4,0.4)	(0.4,0.8)
(α_1, α_2)=(0,0), n=1000				
MSE$_1$	0.0005423088	0.0005466956	0.0005540193	0.0005466956
MSE$_2$	0.0006409966	0.006026144	0.0007281657	0.00612676
MSE$_3$	0.0004569647	0.0008492482	0.0005540551	0.0007111404
MSE$_4$	13152.35	13152.42	13152.42	13152.34
σ_1	0.02344434	0.02134697	0.02369612	0.02353898
σ_2	0.0254884	0.0322492	0.02316626	0.03379329
σ_3	0.02152069	0.02933811	0.02369689	0.02684679
σ_4	115.456	115.4563	115.4563	115.4559
(α_1, α_2)=(1,1),n=1000				
MSE$_1$	0.0007109861	0.0004816762	0.0005470083	0.000565923
MSE$_2$	0.0007844261	0.0006400196	0.0007156163	0.0007034019
MSE$_3$	0.000716252	0.0005325047	0.0005703643	0.0005221559
MSE$_4$	9.024703	27.27848	43.8723	108.5274
σ_1	0.02684388	0.02209492	0.02354571	0.02394933
σ_2	0.02819621	0.02546897	0.02693115	0.02670032
σ_3	0.02694311	0.02323146	0.02404313	0.02300461
σ_4	3.024344	5.258052	6.668221	10.4878
(α_1, α_2)=(2,2),n=1000				
MSE$_1$	0.0005732455	0.0008034585	0.0008714643	0.0008148527
MSE$_2$	0.0007432454	0.0008845181	0.0009294019	0.000870961
MSE$_3$	0.0007161854	0.0008141381	0.0008499219	0.0007437646
MSE$_4$	1.419634	5.58002	13.52397	41.1753
σ_1	0.02410378	0.02853622	0.02971937	0.02873785
σ_2	0.02744612	0.02994113	0.03069139	0.02971078
σ_3	0.02694186	0.02872525	0.02934974	0.0274557
σ_4	1.199507	2.378114	3.702259	6.46001

Table 2. Comparative table of results simulations $n = 1000$

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